GEOPHYSICAL RESEARCH, 2019, vol. 20, no. 1, pp.5-12. https://doi.org/10.21455/gr2019.1-1

UDC 551.511.3: 551.509.313.11

THREE-DIMENSIONAL MODEL OF GENERATION OF ATMOSPHERIC INTERNAL GRAVITY WAVES INDUCED BY INHOMOGENEITIES IN GRAVITATIONAL FIELD

L.Kh. Ingel[1,2,3], A.A. Makosko[2,3]

[1] *Research and Production Association Typhoon, Obninsk, Russia*
[2] *Obukhov Institute of Atmospheric Physics, Russian Academy of Sciences, Moscow, Russia*
[3] *Interdepartmental Center of Analytical Research in Physics, Chemistry and Biology at the Presidium of the Russian Academy of Sciences, Moscow, Russia*

Abstract. In modern models of geophysical fluid dynamics, the gravitational field is usually taken uniform and defined by the single parameter. It is known, however, that the average gravitational force at the earth's surface is superimposed upon by a broad spectrum of gravitational force anomalies (GFAs). This is due mainly to inhomogeneities of the distribution of mass in the Earth's crust. Variations in the gravitational force are certainly very small in magnitude compared to the average value. It is important, however, that such inhomogeneities generate a gravitational-force component tangential to earth's ellipsoid. In plane mesoscale models using Cartesian coordinates (an f-plane or a β-plane), this means that additional volume inhomogeneous forces with a horizontal component have to be taken into account. The dynamics of the atmosphere is quite sensitive to such components.

Recently we showed that in the highly anomalous regions GFAs, in principle, can lead to appreciable dynamic effects, in particular, the generation of regular currents and internal gravity waves (IGW). But this analysis has so far been limited to two-dimensional problems (that is, the effects of two-dimensional GFAs were considered). In this paper, the next step is taken: in the linear approximation, IGW generation in the atmosphere is analytically studied under the action of three-dimensional GFAs on the atmospheric flow above a flat horizontal underlying surface. The terms in the expressions obtained for velocity components and pressure perturbations can be divided into two categories. One of them directly describes flow around equipotential surfaces. These terms do not contain waves propagating with vertical component and slowly decay with altitude on the same scales as the gravity anomaly. Other terms describe internal gravity waves, whose phase velocity is directed downward and the group velocity, upward. The amplitude of these waves in the velocity field exponentially increases with altitude.

Taking into account the three-dimensional geometry of GFAs in the three-dimensional formulation can lead to a noticeable change in results in comparison with the two-dimensional model considered earlier. In addition to the appearance of horizontal motions perpendicular to the background flow, the wavelength and the vertical flux of wave energy can markedly vary: GFAs elongated along the stream can lead to smaller perturbations in amplitude than the "ridge" oriented perpendicular to the background flow. The analytical expression is derived; it shows that the mentioned energy flow is proportional to the background buoyancy frequency, to the squares of the GFAs amplitudes, and to the background flow velocity. According to numerical estimates, this flow can be noticeable, although it is usually much inferior to IGW sources associated with the relief.

Keywords: anomalies of gravity, atmospheric disturbances, three-dimensional analytical model, internal gravity waves.

Authors showed in the recent work [*Ingel, Makosko,* 2017a] that impact of gravity anomalies on the horizontal wind can possibly lead to the generation of internal gravity waves (IGW) in the atmosphere. In this case, the simple two-dimensional analytical model was considered and the theoretical estimates of characteristics of such waves were carried out. The current work presents the next step: the internal gravity waves generated on the three-dimensional gravity anomalies are considered in the three-dimensional formulation.

We shall use the Cartesian coordinate system where z-axis is directed vertically upwards, x-axis is horizontal along the background wind which velocity $\bar{u}$ is assumed to be constant, y-axis is a horizontal axis oriented transversally to the flow. To generalize the standard dynamic equations taking into account the gravity anomalies, we introduce the additional forces (accelerations) $g_x(x,y,z)$, $g_y(x,y,z)$, $g_z(x,y,z)$ – horizontal and vertical components of gravity anomaly [*Ingel, Makosko,* 2017a–c]. From the properties of the gravitational potential follow the equations of "cross" derivatives

$$\frac{\partial g_x}{\partial z} = \frac{\partial g_z}{\partial x} \text{ etc.} \tag{1}$$

Below we will consider the reference system moving along with the horizontal current. In such reference systems components of gravity anomaly also depend on time. Taking into account the considered generalization, the initial linearized system of hydrodynamic equations [*Gill,* 1986, section 6.4] in the reference system moving along the flow, has a form

$$\begin{gathered} \bar{\rho}\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \bar{\rho} g_x\left(x+\bar{u}t, y, z\right), \\ \bar{\rho}\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \bar{\rho} g_y\left(x+\bar{u}t, y, z\right), \\ \bar{\rho}\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - g\rho + \bar{\rho} g_z\left(x+\bar{u}t, y, z\right), \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad \frac{\partial \rho}{\partial t} + w\frac{\partial \bar{\rho}}{\partial z} = 0. \end{gathered} \tag{2}$$

Here t is time; u, v, w are the perturbations of velocity components along x, y, z axes, respectively; $\bar{\rho}(z)$ is a background (unperturbed) air density; p, ρ are the pressure and density perturbations, respectively. As well as in [*Gill,* 1986; *Ingel, Makosko,* 2017a,c], at this stage of the research the Coriolis accelerations are not taken into account (we consider disturbances of rather small horizontal scales).

At the lower boundary of z we assume that the impermeability condition w=0 is satisfied (when considering the processes above the water surface, generally speaking, another condition is required [*Ingel, Makosko,* 2017a, b]).

Similarly, [*Ingel, Makosko,* 2017a], it is easy to get equalities

$$\frac{\partial^2 w}{\partial z \partial t} = \frac{1}{\bar{\rho}}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) - \frac{\partial g_x}{\partial x} - \frac{\partial g_y}{\partial y}, \tag{3}$$

$$\frac{\partial^2 w}{\partial t^2} + N^2 w = -\frac{1}{\bar{\rho}}\frac{\partial^2 p}{\partial z \partial t} + \frac{\partial g_z}{\partial t}, \tag{4}$$

where the square of the buoyancy frequency (Brunt–Väisälä frequency) is

$$N^2 = -\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz}. \tag{4a}$$

Excluding from the system (2) all unknowns (except for w), we arrive at the equation

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(\bar{\rho}\frac{\partial w}{\partial z}\right)\right] + N^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{N^2}{g}\frac{\partial}{\partial t}\left(\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y}\right). \tag{5}$$

Distribution of the background density is approximated by the exponent

$$\bar{\rho}(z) = \rho_0 \exp\left(-\frac{z}{H}\right),$$

from which

$$N^2 = \frac{g}{H}, \tag{6}$$

where the effective medium thickness is H=const.

It is convenient to analyze the model with sinusoidal dependence of gravity anomalies on horizontal coordinates. In the rest reference system, the equations for accelerations g_x, g_y, g_z have the following form:

$$g_x = G\exp(-kz)\cos(k_x x)\sin(k_y y),$$
$$g_y = G\frac{k_y}{k_x}\exp(-kz)\sin(k_x x)\cos(k_y y), \qquad (7)$$
$$g_z = -G\frac{k}{k_x}\exp(-kz)\sin(k_x x)\sin(k_y y),$$

where G is the amplitude; k_x^{-1}, k_y^{-1} are horizontal scales of anomaly; $k = \left(k_x^2 + k_y^2\right)^{1/2}$. In the moving reference system associated with the flow,

$$g_x = G\exp(-kz)\cos\left[k_x(x+\bar{u}t)\right]\sin(k_y y) \text{ etc.}$$

Solution of the equation (5) in this system is sought as

$$w = \left\{W_1(z)\cos\left[k_x\left(x+\bar{u}t\right)\right] + W_2(z)\sin\left[k_x\left(x+\bar{u}t\right)\right]\right\}\sin(k_y y). \qquad (8)$$

For amplitude $W_1(z)$ we get the equation

$$\frac{d^2W_1}{dz^2} - \frac{1}{H}\frac{dW_1}{dz} + \left[\left(\frac{k}{k_x}\frac{N}{\bar{u}}\right)^2 - k^2\right]W_1 = \left(\frac{k}{k_x}\right)^2\frac{N^2G}{\bar{u}g}\exp(-kz); \qquad (9)$$

for amplitude $W_2(z)$ – similar homogeneous equation.

In some cases, it is more convenient to use variables

$$\tilde{W}_{1,2}(z) = W_{1,2}(z)\exp\left(-z/2H\right), \qquad (10)$$

for which the equation (9) has the form

$$\frac{d^2\tilde{W}_1}{dz^2} + \frac{\tilde{W}_1}{\lambda^2} = \left(\frac{k}{k_x}\right)^2\frac{N^2G}{\bar{u}g}\exp(-\tilde{k}z). \qquad (11)$$

Here $\tilde{k} = k + 1/2H$; λ is a vertical scale defined as

$$\lambda = \left[\left(\frac{k}{k_x}\right)^2\left(\frac{N}{\bar{u}}\right)^2 - \left(\frac{1}{2H}\right)^2 - k^2\right]^{-1/2}. \qquad (12)$$

In case of gravity anomalies of sufficiently small spatial scales along the background flow (i.e. at sufficiently large values of k_x) in (12) the expression in square brackets is negative corresponding to the negative values of λ^2. In this case, the solution decays exponentially with the altitude ("trapped waves"). In this paper we consider the case of positive values of λ^2.

We take the characteristic values of atmosphere N=10^{-2} s^{-1}, $\bar{u}$ =10 m/s. Then H=10^5 m, $\bar{u}/N$ =10^3 m, and at horizontal scales of k_x^{-1}, much bigger than a kilometer, we have

$$\lambda \approx \frac{k_x}{k}\frac{\bar{u}}{N}. \qquad (13)$$

General solution of the equation (11) can be written as

$$\tilde{W}_1(z) = C_1\sin\left(\frac{z}{\lambda}\right) + C_2\cos\left(\frac{z}{\lambda}\right) + \left(\lambda\frac{k}{k_x}\right)^2\frac{N^2G}{a\bar{u}g}\exp(-\tilde{k}z),\ a \equiv 1 + (\lambda\tilde{k})^2 \approx 1;$$

similarly – $\tilde{W}_2(z) = C_3 \sin\left(\frac{z}{\lambda}\right) + C_4 \cos\left(\frac{z}{\lambda}\right)$, where C_i are integration constants that are selected considering both the boundary condition on the surface z=0 and radiation condition – the group velocity must be directed upwards [*Gill*, 1986, p.175]. The latter condition corresponds with the downward phase velocity of internal gravity waves [*Gill*, 1986].

In this case we get

$$C_1 = C_4 = 0,$$

$$C_3 = -C_2 = \left(\lambda \frac{k}{k_x}\right)^2 \frac{N^2 G}{a\bar{u}g}.$$

As a result, the vertical velocity is determined by the expression

$$w = W_0 \left\{ \exp(-kz)\cos\left[k_x\left(x + \bar{u}t\right)\right] - \exp\left(\frac{z}{2H}\right)\cos\left[\frac{z}{\lambda} + k_x\left(x + \bar{u}t\right)\right]\right\}\sin(k_y y), \tag{14}$$

where W_0 is the introduced characteristic scale of the amplitude of emerged vertical movements, that has the form

$$W_0 \equiv \left(\lambda \frac{k}{k_x}\right)^2 \frac{N^2 G}{a\bar{u}g} \approx \bar{u}\frac{G}{g}.$$

Compared with the two-dimensional problem [*Ingel, Makosko*, 2017a], in addition to the appearance of the dependence on the y-coordinate transverse to the flow, combination $N\left(k/k_x\right)$ appears in (14) instead of buoyancy frequency N.

In other words, when shifting to the three-dimensional problem the "effective stratification" is amplified in the expression for the vertical velocity. According to (13), the vertical component of the wave vector λ^{-1} can be significantly larger than in the case of the two-dimensional problem.

Substituting (14) into (3), it is easy to get the expression for the pressure deviation –

$$p = \bar{\rho}\frac{G}{k_x}\Big\langle \left(1 - k\lambda\varepsilon\right)\exp(-kz)\sin\left[k_x\left(x + \bar{u}t\right)\right] -$$
$$- \varepsilon\left\{\frac{\lambda}{2H}\sin\left[\frac{z}{\lambda} + k_x\left(x + \bar{u}t\right)\right] + \cos\left[\frac{z}{\lambda} + k_x\left(x + \bar{u}t\right)\right]\right\}\exp\left(\frac{z}{2H}\right)\Big\rangle\sin(k_y y), \tag{15}$$

where the dimensionless parameter $\varepsilon \equiv \lambda N^2/ag \approx \left(k_x/k\right)N\bar{u}/g$.

Now, using the first two equations from (2), it is possible to express in the explicit form the perturbation of the horizontal velocity:

$$u = U\Big\langle k\lambda e^{-kz}\sin\left[k_x(x + \bar{u}t)\right] +$$
$$+ \left\{\frac{\lambda}{2H}\sin\left[\frac{z}{\lambda} + k_x(x + \bar{u}t)\right] + \cos\left[\frac{z}{\lambda} + k_x(x + \bar{u}t)\right]\right\}e^{z/2H}\Big\rangle\sin(k_y y),$$

$$v = U\frac{k_y}{k_x}\Big\langle -k\lambda e^{-kz}\cos\left[k_x(x + \bar{u}t)\right] + \tag{16}$$
$$+ \left\{-\frac{\lambda}{2H}\cos\left[\frac{z}{\lambda} + k_x(x + \bar{u}t)\right] + \sin\left[\frac{z}{\lambda} + k_x(x + \bar{u}t)\right]\right\}e^{z/2H}\Big\rangle\cos(k_y y),$$

where was introduced the scale of horizontal velocity perturbations

$$U = \frac{\lambda N^2 G}{ag\bar{u}k_x} \approx \frac{NG}{kg}.$$

Terms in these expressions can be divided into two categories. First category includes terms with multiplier exp(–*kz*), directly describing the flow around equipotential surfaces. They are not wave-like (do not contain waves propagating vertically) and slowly decay with altitude on the same scales k^{-1}, as the gravity anomaly.

Terms of the second category contain multipliers $\sin\left[z/\lambda + k(x+\bar{u}t)\right]$, $\cos\left[z/\lambda + k(x+\bar{u}t)\right]$ and describe internal gravity waves, whose phase velocity is directed downwards and the group velocity, upwards. The amplitude of these waves in the velocity field exponentially increases with altitude as exp(*z/*2*H*). The vertical component of the wave vector is λ^{-1}, frequency is $\omega = k\bar{u}$, the vertical phase velocity is $k\lambda\bar{u}$.

The solution depends on several spatial scales – k_x^{-1}, k_y^{-1}, H, $\bar{u}/N$. The order of at least one of the first two determining the spatial scales of the anomaly is considered to be not less than 100 km.

The order of the third, apparently, should be considered close to the thickness order of the troposphere, i.e. ~10 km. (Typical value for the atmosphere N=10^{-2} s^{-1} is obtained from (4a) with the characteristic vertical scale of change of background density H of the order of 100 km, while in the real atmosphere H~10 km. The noted is a side-effect of using the incompressible fluid model.)

The fourth scale with the considered values of parameters is no more than several kilometers. From such ratio of scales follows (13), as well as

$$\lambda << H << k^{-1}, \tilde{k} \approx 1/2H, \lambda\tilde{k} \approx \lambda/2H \approx k_x\bar{u}/2kHN << 1, k\lambda << 1, a \approx 1, \varepsilon \approx \frac{\lambda}{H} << 1. \quad (17)$$

It should be noted that vertical component of the wave vector in the three-dimensional problem, basically, can be significantly larger than in the two-dimensional (if anomalies are elongated along the background flow due to k_x/k factor in (13)). Unlike the two-dimensional problem the velocity component v along the transverse y-axis appears at a flow around the three-dimensional anomaly. With the increase in k_y , its amplitude $GNk_y/agkk_x$ increases monotonically from zero (two-dimensional problem) to the values of the order GN/gk_x at k_y, which order is greater than k_x order or close to it.

Note, that

$$GN/gk_x \sim Nh, \quad (18)$$

where h is deviation amplitude of geoid in the considered anomaly. Expression (18) for the amplitude of horizontal velocity deviations was already mentioned in the previous works of the authors (see, for example, [*Ingel, Makosko,* 2017c]) in connection with another perturbation mechanisms associated with gravity anomalies and, apparently, is a very common pattern.

The vertical flow of the wave energy is estimated by the formula

$$F_z = \overline{pw},$$

where the bar indicates the averaging by the wave length [*Gill*, 1986]. It can be seen, that with such averaging the non-zero contribution to (14) and (15) is made only by the product of terms with $\cos\left[z/\lambda + k(x+Ut)\right]$. It is easy to obtain approximate expression independent of the altitude

$$F_z \approx \frac{1}{2}\rho_0 \frac{N\bar{u}^2 G^2}{g^2 k} \sin^2(k_y y). \tag{19}$$

This result differs from the corresponding formula of the two-dimensional problem [*Ingel, Makosko,* 2017a] by the presence of multiplier $\sin^2(k_y y)$ and by k instead of k_x in the denominator. The latter means that the wave energy flow in the three-dimensional problem, generally speaking, can significantly decrease as compared to the two-dimensional (at $k_y \geq k_x$). This is understandable, since anomalies elongated along the stream lead to smaller perturbations in amplitude than the "ridge" oriented perpendicular to the background flow. And otherwise the result (19), as explained in [*Ingel, Makosko,* 2017a], is explicitly interpreted. The horizontal air flow in the gravity anomalies is curved and tends to move along the surfaces of equal potential [*Ingel, Makosko,* 2017b] (the situation is more complicated near the underlying surface since the impermeability condition excludes the possibility of existence of a motion component normal to this surface). This means the appearance of the vertical component of velocity with the amplitude $w \sim uG/g$. The last expression taking into account (13) and (17) is approximately equal to the amplitude of (14). Amplitude of the vertical displacement of the air flow (or equipotential surfaces – deviation amplitude of geoid) h of order G/gk_x, from where $G/g \sim hk_x$. This shows, that (19) is very close (with an accuracy to notations) to the formula (6.8.7), given in [*Gill*, 1986], that describes the vertical energy flux of internal gravity waves, caused by inhomogeneities in the relief with amplitude h. Thus, when we take into account the impact of gravity anomalies (curved equipotential surfaces) it leads to the effect similar to the influence of relief inhomogeneities of the same amplitude and horizontal scales.

If we take the amplitude of gravity anomaly G=10^{-3} m/s^2, $k_x = k_y = 2\cdot 10^{-5}$ m^{-1} (that corresponds to the length of a half-wave in each direction of about 150 km), $\bar{u}$ =20 m/s, N=10^{-2} s^{-1}, ρ_0=1 kg/m^3, then we get u=v~0.03 m/s; energy flux of 10^{-3} W/m^2. For comparison note that according to [*Jarvis*, 2001], the average energy flux coming from the lower atmosphere to the upper due to the wave disturbances and tidal oscillations, is about $2\cdot 10^{-4}$ W/m^2.

Thus, on the basis of the three-dimensional model in linear approximation analytical solutions were obtained showing that gravity anomalies when taking into account the three-dimensional geometry can, generally speaking, significantly affect the characteristics of generated internal gravity waves in the atmosphere. In addition to the appearance of motions perpendicular to the background flux, the wavelength and the vertical flux of the wave energy can vary considerably.

This work was supported by the Program №51 of the fundamental research of the Presidium of the RAS.

References

Gill A.E. *Atmosphere-Ocean Dynamics,* New York: Academic Press, 1982.

Ingel L.Kh. and Makosko A.A. Generation of atmospheric internal waves induced by inhomogeneities in gravity field, *Geofizicheskie Issledovaniya* (Geophys. Res.), 2017a, vol. 18, no. 3, pp. 60-66.

Ingel L.Kh. and Makosko A.A. Geostrophic flow disturbances generated by inhomogeneities of the gravitational field, *Izv. Atmos. Ocean. Phys*., 2017b, vol. 53, no. 5, pp. 508-515.

Ingel L.Kh. and Makosko A.A. On one mechanism of gravity field inhomogeneities influence on atmosphere dynamics, *Technical Physics*, 2017c, vol. 62, no. 9, pp. 1322-1326.

Jarvis M.J., Atmospheric science: Bridging the atmospheric divide, *Science*, 2001, vol. 293, no. 5538, pp. 2218-2219.