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ISOSTATIC RECOVERY FLOWS IN THE LITHOSPHERE

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Abstract. Laboratory experiments with rock samples show that their creep at small strains is transient. Therefore we can assume that the lithospheric plates, where strains are small, demonstrate the transient creep that is described by the linear hereditary Andrade rheological model. The effective viscosity that characterizes the transient creep is lower than the effective viscosity at steady-state creep and depends on the characteristic time of the process. The typical duration of isostatic rebound after an initial disturbance of the Earth's surface is several thousand years, and therefore, the depth distribution of the rheological properties of the lithosphere and crust is different from the distribution that corresponds to slow geological processes.

It is shown that when considering the isostatic recovery process, the upper crust can be modeled as a thin elastic plate and the underlying lower crust and lithosphere, as a half-space with transient creep. For such a system, the continuum mechanics equations are solved using the Fourier and Laplace transforms. The solutions are obtained in the form of transverse waves that propagate, with strong attenuation, from the area of the initial disturbance along the Earth's surface and cause its vertical displacements. These solutions, called the inertialess Rayleigh waves, depend on the initial disturbance. In the case of a point initial disturbance, the analytical expression is found for these waves that gives an explicit dependence of the vertical displacements of the Earth's surface on the horizontal coordinates and time. The inertialess Rayleigh waves can be regarded as a mechanism of modern vertical movements of the crust.

Keywords: transient creep, isostatic recovery, vertical movements of the Earth's surface.

Introduction

Laboratory experiments with rock samples show that when creep strains are small, transient creep takes place: at constant applied stresses, the creep strain rate decreases with time. Plate tectonics allows only small deformations in lithospheric plates and, therefore, the creep of the lithosphere is transient. Hereinafter, the term "lithosphere" means a lithospheric plate, and the boundaries between the plates, where the deformations are large, are excluded from consideration. The lithospheric plate is a cold boundary layer formed by mantle convection, and the thickness of continental plates may exceed 200 km. Deformations of the lithosphere consist of creeping, elastic, and brittle (pseudo-plastic) deformations. These rheological mechanisms compete with each other, and the mechanism that corresponds to the lowest effective viscosity dominates. The assumption of the steady-state creep of the lithosphere that is commonly used in geophysical studies, leads to a very high effective viscosity characterizing creep at small deformations. In this case, the rheology of the lithosphere can be described by an elastic-brittle rheological model neglecting creep. Transient creep corresponds to a much lower effective viscosity than steady-state creep. Therefore, transient creep must be taken into account when considering geophysical processes in the lithosphere. The effective viscosity corresponding to transient creep depends on the characteristic duration of the geophysical process under consideration. *Birger* [1998, 2000, 2012, 2013] considered the process of formation of sedimentary basins on continental cratons. The characteristic time of this process is about 10^8 years. In the present study we consider the recovery of isostatic equilibrium after the initial small-scale perturbation of the relief of the earth's surface (the horizontal dimension of the perturbed region does not exceed 1000 km). As a result of the recovery process, the Earth's surface returns to a flat position that corresponds to the equilibrium state with a uniform distribution of density in the horizontal direction. The characteristic duration of this process does not exceed 1000 years and, therefore, the distribution of rheological properties over the depth of the lithosphere and the crust differs from that corresponding to slower processes. The process of isostatic recovery is accompanied by inertialess Rayleigh waves that, strongly damping, propagate from the region of the initial perturbation along the Earth's surface and cause its vertical displacements

Rheological model

Transient creep characteristic for sufficiently small deformations, is described by the Boltzmann integral equation

$$\varepsilon_{ij}(t) - \varepsilon_{ij}(0) = \frac{1}{2}\int_0^t K(t_1)\sigma_{ij}(t - t_1)dt_1\,, \qquad (1)$$

where ε_{ij} is the tensor of deviatoric creep strains; σ_{ij} is the tensor of deviatoric stresses, *K (t)* is the integral creep kernel

$$K(t) = t^{-2/3}/3A\,, \qquad (2)$$

In the case when a constant stress σ_{ij}, is applied at time t = 0, the relationship (1) and (2) is reduced to the Andrade law that describes the transient creep of rocks with sufficiently small strains and constant stresses:

$$2\varepsilon_{ij} = \sigma_{ij} t^{1/3}/A\,, \qquad (3)$$

where $\varepsilon < \varepsilon_{tr}$, $\varepsilon = \left(\varepsilon_{ij}\varepsilon_{ij}/2\right)^{1/2}$, $\varepsilon_{tr} \approx 10^{-3} - 10^{-2}$, ε is the second invariant of the deviatoric strain tensor and ε_{tr} is its transition value.

The medium described by the rheological equations (1) and (2) will be called the Andrade medium. For sufficiently large strains $\varepsilon > \varepsilon_{tr}$ transient creep is replaced by steady-state creep that is described by the rheological model of a power non-Newtonian fluid. *Birger* [1998] introduced a nonlinear hereditary rheological model and considered the situation when the flow associated with small deformations is superimposed on the stationary flow described by the rheological model of a power non-Newtonian fluid. In this paper, it is shown that the Andrade linear integral model is applicable if the characteristic duration of the flow under consideration is less than $\varepsilon_{tr}/\dot{\varepsilon}$, where $\dot{\varepsilon}$ is the rate of strain in the stationary flow. Since $\dot{\varepsilon} \approx 10^{-15}$ s^{-1} in the mantle beneath the lithosphere [*Turcotte, Schubert*, 1985], the Andrade model is applicable if the duration of the superimposed flow does not exceed 10^{12} s ≈ $3{\cdot}10^4$ years.

The value of the Andrade rheological parameter depends on temperature (there is a significant vertical temperature gradient in the lithosphere) and mineralogical composition. The distribution of the Andrade parameter over the depth of the lithosphere was obtained in [*Birger*, 2013]. In the upper crust with a thickness of about 20 km, this parameter decreases with depth from the value A ≈10^{16} Pa·s$^{1/3}$ to A ≈10^{13} Pa·s$^{1/3}$. In the lithosphere lying under the upper crust, the average value of the Andrade parameter is estimated as $A \approx 5{\cdot}10^{12}$ Pa·s$^{1/3}$. In the mantle under the lithosphere, the average value of the Andrade parameter is an order of magnitude higher, which is due to higher pressure [*Birger*, 2013].

Since the total deviatoric deformation of the medium can be expressed as the sum of the deviatoric deformation of creep (1) and the deviatoric deformation of elasticity

$$\varepsilon_{ij} = \sigma_{ij} / 2\mu , \tag{4}$$

where μ is an elastic shear modulus, the elastic-creeping medium with transient creep is described by equation

$$\varepsilon_{ij}(t) - \varepsilon_{ij}(0) = \frac{\sigma_{ij}}{2\mu} + \frac{1}{2}\int_0^t K(t_1)\sigma_{ij}(t - t_1)dt_1 . \tag{5}$$

Using the Laplace transform, the rheological equations (1) and (2) can be written in the form

$$\sigma^*_{ij} = 2G^*_A\varepsilon^*_{ij}, \quad G^*_A = 1/K^* = As^{1/3}\Gamma(1/3)/3 \approx As^{1/3} , \tag{6}$$

where the Laplace transform is marked by the asterisk, G^*_A is the Laplace analog of the shear modulus for the Andrade medium, s is the Laplace variable, and $\Gamma(1/3) \approx 3$ is the gamma-function. Equation (5) corresponds to the Laplacian image

$$\sigma^*_{ij} = 2G^*\varepsilon^*_{ij}, \quad G^* = \frac{\mu As^{1/3}}{\mu + As^{1/3}} , \tag{7}$$

where G^* is the Laplace analog of the shear modulus for an elastic-creeping medium.

As follows from (7), the elasticity of the medium can be neglected if

$$s << (A/\mu)^3 . \tag{8}$$

Under this condition, the Laplace analog of the shear modulus G^*_A for the Andrade medium is significantly smaller than the elastic shear modulus μ. According to the Laplace transform property, it follows from (8) that

$$t >> (A/\mu)^3 . \tag{9}$$

The compressibility of the medium can be neglected if the Laplace analog of the shear modulus G^*_A for the Andrade medium is significantly smaller than the bulk modulus K. Since the elastic shear modulus μ is smaller than the bulk modulus ($K \approx 3\mu$), the condition (9) allows neglecting not only the elasticity but also the compressibility of the medium.

The elastic modulus of the Earth's crust is estimated as $\mu \approx 5 \cdot 10^{10}$ Pa. In the upper crust, the Andrade rheological parameter is estimated as $A \approx 10^{14}$ Pa·s$^{1/3}$. Since the characteristic time of the process under consideration does not exceed 1000 years, with such a large value of the Andrade parameter in the upper crust

$$t << (A/\mu)^3 ,$$

and the upper crust behaves as an elastic medium. More precisely, the upper crust, whose thickness is of the order of 20 km, is brittle - elastic, and elasticity dominates only in the lower layer of the upper crust, whose thickness is about 10 km. The uppermost layer of the crust is dominated by brittleness and its strength is very small [*Birger*, 2013]. Hereinafter, we simply neglect the presence of a brittle layer and assume that the upper crust is an elastic layer 10 km thick. For the lower crust and mantle lithosphere, average depth value of the rheological parameter is estimated as $A \approx 5 \cdot 10^{12}$ Pa·s$^{1/3}$ and the right side of inequality (9) is 10^8 s ≈ 3 years. Thus, the lithosphere under the elastic upper crust behaves as the creeping

Andrade medium, without showing elasticity and compressibility for times in excess of 10 years.

The influence of inertia is negligibly small (the inertia forces are small in comparison with the forces arising at deformations of the Andrade medium), if

$$t >> \left(\rho L^2/A\right)^{3/5}, \tag{10}$$

where L is the characteristic length scale. When $L \leq 1000$ km, the right-hand side of inequality (10) does not exceed 100 s and, neglecting elasticity, the more we can neglect inertia. Process with a typical time of about 1000 years is slow enough to neglect elasticity, compressibility, and inertia, but it is not slow to take into account the buoyancy Archimedes force due to the vertical temperature gradient in the lithosphere.

If we take into account the elasticity of the lithosphere under the upper crust and the inertia, the perturbation of the relief of the Earth's surface causes not only inertialess Rayleigh waves, but also ordinary Rayleigh waves that rapidly propagate and rather weakly attenuate.

Statement of the problem. Fourier and Laplace transforms

In this paper, two-dimensional statement of the problem will be used, which is explained by two reasons. First, numerous studies of postglacial flows carried out within the framework of the Newtonian rheological model show that the transition from 2D to 3D formulation does not change the results significantly. The existing observational data are not so definite as to seek the solution with an accuracy exceeding the estimate by an order of magnitude. Secondly, this study deals with relatively narrow areas of initial perturbations of the relief. When the melting of ice occurs on huge glacial areas, such as in Fennoscandia and Canada, glacial loads are first removed in narrow elongated areas at the edges of large glaciations. The data on small-scale postglacial uplifts refer primarily to such highly elongated regions of initial disturbances that cause flows that can be described within the frames of 2D models.

We consider an elastic thin plate lying on a half-space with Andrade rheology. The origin is placed on the upper surface and the z axis is directed vertically upwards. A thin plate (z = 0) models the upper elastic crust and the half-space (z<0), the underlying lithosphere and mantle. Equations of equilibrium for an incompressible half-space are written in the form

$$-\frac{\partial p}{\partial x}+\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xz}}{\partial z}=0, \tag{11}$$

$$-\frac{\partial p}{\partial z}+\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{zz}}{\partial z}=0, \tag{12}$$

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}=0, \tag{13}$$

where p is the pressure perturbation; u_x and u_z are the displacements, and equation (13) is the incompressibility condition. Equations (11) - (13) are added to the original rheological equation (1), where the strains are related to the displacements by the relations

$$\varepsilon_{xx}=\frac{\partial u_x}{\partial x}, \qquad \varepsilon_{zz}=\frac{\partial u_z}{\partial z}, \qquad \varepsilon_{xz}=\frac{1}{2}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right). \tag{14}$$

The boundary conditions at the upper surface of the half-space (z = 0) are determined by the force action of the elastic plate

$$\frac{2}{1-\nu}\frac{\partial^2 u_x}{\partial x^2}+\frac{1}{\mu h}\sigma_{xz}=0\,, \tag{15}$$

$$N\frac{\partial^4 u_z}{\partial x^4}+\rho g u_z+\sigma_{zz}-p=0\,, \tag{16}$$

$$N=\frac{\mu h^3}{6(1-\nu)}\,, \tag{17}$$

where $\nu$is the Poisson's ratio, N is the flexural rigidity of the elastic plate with a thickness h. The displacements u_x and u_z in the plate are equal to the displacements in the underlying half-space at $z = 0$.

The physical variables in equations (1) and (11) - (14) depend on the horizontal coordinate x, the vertical coordinate z, and the time t. Applying the Fourier transform with respect to the coordinate x and the Laplace transform with respect to time t to these equations and excluding all physical variables excepting the vertical displacement, we get the relations

$$U_x^*=\frac{i}{k}DU_z^*\,, \tag{18}$$

$$P^*=\frac{G(s)}{k^2}(D^3-k^2D)\left(U_z^*-\frac{U_0}{s}\right), \tag{19}$$

$$\Sigma_{zz}^*=-\Sigma_{xx}^*=2G(s)D\left(U_z^*-\frac{U_0}{s}\right), \tag{20}$$

$$\Sigma_{xz}^*=\frac{i}{k}G(s)(D^2+k^2)\left(U_z^*-\frac{U_0}{s}\right), \tag{21}$$

where the differential operator $D=d/dz$ is introduced, and $U_0=U_0(x,z)$ is the initial ($t = 0$) distribution of vertical displacements. In the equations (18) - (21), the Fourier transforms of the physical variables are denoted by corresponding capital letters, the Laplace images are marked with the asterisk, k is the wave number (the Fourier variable), and s is the Laplace variable. Then we obtain the ordinary differential equation for the vertical displacement

$$(D^2-k^2)^2\left(U_z^*-\frac{U_0}{s}\right)=0\,. \tag{22}$$

The solution of equation (22) that satisfies the boundary condition for $z\to-\infty$ is written as

$$U_z^*-\frac{U_0}{s}=\left[C_1(s)+C_2(s)z\right]\exp\left(|k|z\right); \tag{23}$$

where C_1 and C_2 are arbitrary integration constants that depend on the Laplace variable and the wave number k can take negative values. Substitution of the solution (23) into the boundary conditions (15) - (16) allows us to eliminate arbitrary constants and represent the vertical displacement of the upper surface ($z = 0$) in the form

$$U_z^*=\frac{U_0(k)}{s}\cdot\frac{2|k|G_A^*(s)}{\rho g+Nk^4+2|k|G_A^*(s)}\,, \tag{24}$$

where $G_A^*(s)=As^{1/3}$ is the analog of the shear modulus for the Andrade medium. It should be noted that the substitution of (18) - (23) into the boundary condition (15) leads to the relation

$C_2 = -C_1/k$, at which horizontal displacements and tangential stresses are absent on the upper surface.

In order to find the asymptotic (small times) dependence of the Laplace origin on time, it is sufficient to expand the Laplace image in a series in powers of s in a neighborhood of $s = \infty$ and invert by Laplace each term of the series [*Doetsch*, 1967]. The right-hand side of (24) for $s\to\infty$ can be expressed in the form of a series

$$U_z^* = U_0(k)\left(s^{-1} - \Phi(k)s^{-4/3} + \ldots\right), \tag{25}$$

$$\Phi(k) = \frac{\rho g + Nk^4}{2A|k|}. \tag{26}$$

Inverting the terms of the series (25), we obtain the asymptotic dependence of the vertical displacement on time

$$U_z(t) = U_0(k)\left(1 - \Phi(k)\frac{t^{1/3}}{\Gamma(4/3)} + \ldots\right), \tag{27}$$

where the gamma-function at the point 4/3 is estimated as

$$\Gamma(4/3) = \frac{1}{3}\Gamma(1/3) \approx 1. \tag{28}$$

The asymptotic dependence (27) is valid if

$$t << \left(\Phi(k)\right)^{-3}. \tag{29}$$

As follows from (28) and (29), the asymptotic dependence (27) can be represented in the form

$$U_z(t) = U_0(k)\exp\left(-\Phi(k)t^{1/3}\right). \tag{30}$$

The physical parameters entering into equations (16) and (26) are estimated as

$$\rho\approx3\cdot10^3\ \text{kg/m}^3,\quad g\approx10\ \text{m/s}^2,\ A\approx5\cdot10^{12}\ \text{Pa}\cdot\text{s}^{1/3},\ \mu\approx5\cdot10^{10}\ \text{Pa},\ \nu\approx0.25,\ N\approx10^{22}\ \text{Pa}\cdot\text{m}^3. \tag{31}$$

With these estimates characterizing the elastic upper crust and the underlying lithosphere, the function $\Phi(k)$ determined by (26) takes the form

$$\Phi(k) \approx (3\cdot10^{-9} + 10^9 k^4)/|k|. \tag{32}$$

Let at the initial time $t = 0$ the displacement of the upper surface ($z = 0$) is given as

$$u_z = u_0(x). \tag{33}$$

The Fourier transform of (33) is written in the form

$$U_0(k) = \int_{-\infty}^{\infty} u_0(x)\exp(-ikx)dx, \tag{34}$$

and the inverse Fourier transform is

$$u_0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_0(k)\exp(ikx)dk. \tag{35}$$

Inversing the Fourier image (30), we find

$$u_z(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_0(k)\exp\left(-\Phi(k)t^{1/3} + ikx\right)dk\,. \qquad (36)$$

If $u_0(x)$ is an even function, (36) can be written as

$$u_z(x,t) = \frac{1}{\pi}\int_{0}^{\infty} U_0(k)\exp\left(-\Phi(k)t^{1/3}\right)\cos(kx)dk\,. \qquad (37)$$

Let the initial displacement have the form of a "step"

$$\begin{aligned} u_0(x) &= u_0, \quad \text{if} \quad -l/2 < x < l/2, \\ u_0(x) &= 0, \quad \text{if} \quad x < -\tfrac{l}{2} \ \text{or}\ x > \tfrac{l}{2}. \end{aligned}, \qquad (38)$$

The Fourier transform for this "step" has the form

$$U_0(k) = 2u_0\frac{\sin(lk/2)}{k}\,. \qquad (39)$$

For a sufficiently large width l of the initial perturbation, the values of the function (39) are very small when $k<-2\pi/l$ and $k> 2\pi/l$, i.e., the wider the perturbation region, the narrower is the range of wave numbers k, where the Fourier image is different from zero. Thus, the integration on the right-hand side of (36) is carried out over the region $-2\pi/l<k<2\pi/l$ in the neighborhood of the point $k = 0$. According to the solution (23), for a fixed wave number k, the isostatic flow causes displacements in the lithosphere depending on the depth as exp ($-kz$). Since $k<2\pi/l$, we can regard that this flow penetrates into the lithosphere to a depth of the order of l/π. Therefore, the flows arising after the removal of small-scale glaciations or other surface loads (for example, drying salt lakes), for which l does not exceed 200 km, are concentrated in the lithosphere. Caused by large-scale glacial loads ($l \approx 1000 \div 3000$ km), flows that penetrate into the low mantle and recover isostasy over a period of time of about 10000 years, are not considered in this paper.

As follows from (39), the Fourier transform does not depend on k, if the width of the initial perturbation is small ($lk \ll 1$),

$$U_0(k) = u_0 l\,; \qquad (40)$$

The image (40) corresponds to a point initial perturbation

$$u_0(x) = u_0 l\cdot\delta(x), \quad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ikx)dk\,, \qquad (41)$$

where $\delta(x)$ is the delta function.

It is known that the inverse Fourier transform applied to the image of a function, having a discontinuity, leads to an original that differs from the original function at the discontinuity (this distortion is called the Gibbs effect). The "step" (38) has a discontinuity at $x = l$.

Therefore, it is more convenient to use a continuous function to describe the initial perturbation, for example, the Gaussian distribution [*Cathles*, 1975]. If the dependence of the initial perturbation on the horizontal coordinate x is characterized by the Gaussian distribution

$$u_0(x) = u_0 \exp(-bx^2), \tag{42}$$

the characteristic width of the initial perturbation is estimated as $l \approx 1/\sqrt{b}$, and the Fourier transform takes the form

$$U_0(k) = u_0 \frac{1}{2}\sqrt{\frac{\pi}{b}} \exp\left(-\frac{k^2}{4b}\right). \tag{43}$$

To obtain the dependence of the vertical displacements of the surface on the horizontal coordinate and time for a given initial distribution (42), it is sufficient to substitute the image (43) into equation (36) or (37). Figure 1 shows the results of numerical integration. The displacements are presented in dimensionless form $u_z(x,t)/u_0$, where u_0 is the vertical displacement u_0 in the center of the region of the initial disturbance, and the horizontal coordinate is measured in kilometers. Time is counted from the moment of the initial perturbation. The initial displacements are given by the Gaussian distribution and correspond to an isostatically unbalanced depression, the horizontal dimension of which is about 200 km.

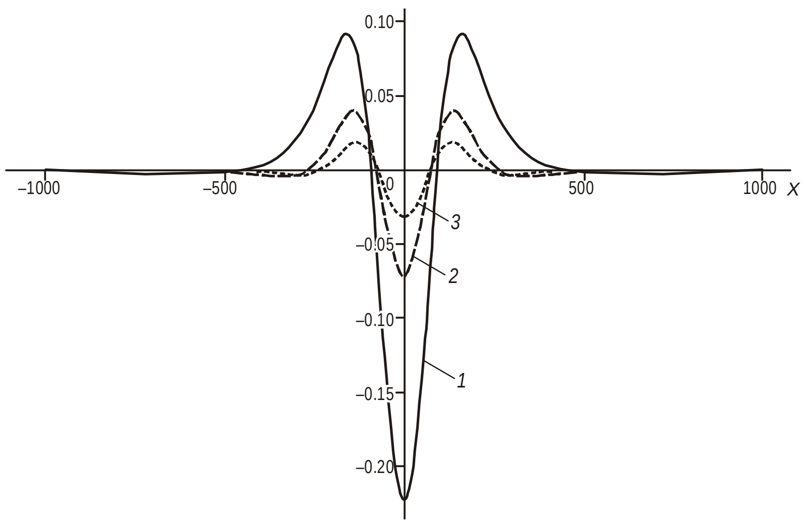

Fig.1. Dependence of the earth's surface vertical displacements on the horizontal coordinate at different moments of time. Curve 1 corresponds to 30 years, curve 2 to 300 years, curve 3 to 1000 years.

Point initial perturbation

In the case of a point initial perturbation (perturbation of any initial width l can be regarded as a point perturbation when we consider displacements at a sufficient distance from the initial perturbation, that is, for $x \gg l$, it is possible to obtain an analytic solution of the problem of vertical surface motions. For the point initial perturbation, the dependence of the vertical displacements of the surface on the horizontal coordinate and time is determined by the integral

$$u_z(x,t) = \frac{u_0 l}{2\pi} I,$$
$$I = \int_{-\infty}^{\infty} \exp\left(-\Phi(k)t^{1/3} + ikx\right)dk. \tag{44}$$

In the integrand there is a function $\Phi(k)$, given by formula (32). This function has a sharp minimum, which is found from condition

$$\frac{d\Phi}{dk} = 0 \tag{45}$$

and is attained at $k=k_m$, where

$$k_m = \left(\frac{\rho g}{3N}\right)^{1/4} \approx 3\cdot 10^{-5} \text{ м}^{-1}. \tag{46}$$

If $k >> k_m$, the values $\Phi(k)$ are very large and the integrand vanishes. Therefore, we can assume that the integration on the right-hand side of (44) takes place in a rather narrow range of wave numbers lying in the neighborhood of $k=k_m$. The neighborhood of the point $k=-k_m$ gives exactly the same contribution to the integral (44).

By (45), the expansion of the function $\Phi(k)$ in a power series in the $k=k_m$ neighborhood has the form

$$\Phi(k) = \Phi_m + a(k-k_m)^2 + \dots, \tag{47}$$

where

$$\Phi_m = \Phi(k_m) \approx 1.3\cdot 10^{-4} \text{ s}^{-1/3},\ a \approx 1.9\cdot 10^5 \text{ m}^2\cdot\text{s}^{-1/3}. \tag{48}$$

The power series (47) representing the function $\Phi(k)$ given by (26) converges if $|k-k_m| \le k_m$, i.e., the radius of convergence of this series is $R=k_m$.
After changing the variable

$$v = k - k_m \tag{49}$$

the integral (43) takes the form

$$I = 2\int_{-\infty}^{\infty} \exp\left(-t^{1/3}(\Phi_m + av^2) + ik_m x + ivx\right)dv. \tag{50}$$

The factor 2 on the right-hand side of (50) appears due to the contribution of the neighborhood of the point $k=-k_m$.

Equation (50) can be rewritten as

$$I = 2E_m \int_{-\infty}^{\infty} \exp\left(t^{1/3} f(v)\right)dv, \tag{51}$$

where

$$E_m = \exp\left(ik_m x - t^{1/3}\Phi_m\right),\quad f(v) = -av^2 + i\frac{x}{t^{1/3}}v. \tag{52}$$

For large values of t, the integral in equation (51) has a pronounced maximum, and therefore the main contribution to the value of this integral is given by the neighborhood of the maximum point. The value of this integral is found by the saddle point method [*Copson*, 1966]. Applying the saddle point method, we assume that the integration variable v is not real, but complex. Since after such a substitution the integrand has no singular points, integration along the real axis can be replaced by integration over another contour in the complex plane.

A contour passing through the saddle point is chosen, in the neighborhood of which the integrand complex function has the most pronounced maximum. The stationary point v_0 is found from condition

$$\frac{df}{dv} = 0 . \tag{53}$$

The function of the complex variable $f(v)$ given by (52) has a stationary point

$$v_0 = i\frac{x}{2at^{1/3}} . \tag{54}$$

As follows from (52) and (54), in a neighborhood of the stationary point v_0 the function $f(v)$ can be represented in the form

$$f(v) = f(v_0) - a(v - v_0)^2, \quad f_0 = f(v_0) = -\frac{x^2}{4at^{2/3}} . \tag{55}$$

Substituting (55) into (51), we obtain

$$I/2 = E_m \exp\left(t^{1/3} f_0\right) \int_{-\infty}^{\infty} \exp\left[-t^{1/3} a(v - v_0)^2\right] dv . \tag{56}$$

For the integrand complex function, the stationary point v_0 is a saddle point. In the saddle-point method, the path of integration on the complex plane is chosen in such a way that it runs through the saddle point, and in a small neighborhood of this point is a straight line segment, in which the difference $f(v) - f(v_0)$ has real negative values vanishing only at $v=v_0$. According to (55), this difference takes the form

$$f(v) - f(v_0) = -a(v - v_0)^2 = -|a| r^2 \exp\left[i(\alpha + 2\beta)\right], \tag{57}$$

where $a = |a| \cdot \exp(i\alpha);\ v - v_0 = r \cdot \exp(i\beta)$.

From the relation (57), it is clear that an integration path on the complex plane can be chosen as a straight line segment, where $\beta = -\alpha/2$. On this segment, the difference takes real negative values, and the integral is written as

$$\int_C \exp\left[-a(v - v_0)^2 t\right] dv = \exp\left(-\frac{i\alpha}{2}\right) \int_{-R}^{R} \exp\left(-|a| r^2 t\right) dr , \tag{58}$$

where R is the radius of convergence of the series (47). The integral on the right-hand side of (58) can be represented as

$$\int_{-R}^{R} \exp\left(-|a| r^2 t\right) dr = \sqrt{\frac{\pi}{|a| t}} \operatorname{erf}\left(R\sqrt{|a| t}\right), \tag{59}$$

i.e., this integral is easily transformed to the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-y^2\right) dy .$$

It is known that $\operatorname{erf}(x) = 1$ at $x >>1$, and at $x >1$ is close to 1.

Thus, as follows from (44), (56), and (58), the required distribution of the vertical displacements of the surface takes the form

$$u_z(x,t) = \frac{2u_0 l}{\pi} \sqrt{\frac{\pi}{at^{1/3}}} \operatorname{erf}\left(R\sqrt{at^{1/3}}\right) \exp\left(ik_m x - t^{1/3}\Phi_m - \frac{x^2}{4at^{1/3}}\right). \tag{60}$$

The found solution (60) is valid for sufficiently long times (from several hundreds to several thousand years). The upper bound on time is imposed by condition (29), where $k=k_m$.
The plots in Fig. 2 are constructed using (60) and show the dependence of the vertical displacements on the horizontal coordinate at different times. Differentiating the right-hand side of (60) with respect to t for a fixed value of x, it is not difficult to find the velocity of the vertical motion of the Earth's surface at points sufficiently far from the region of the initial disturbance of the relief. For example, if $u_0 = 100$ m, $l = 10$ km, $x = 100$ km, the velocity $\frac{du_z}{dt}$ reaches its maximum value (about 1 mm / year) during the time $t \approx 600$ years.

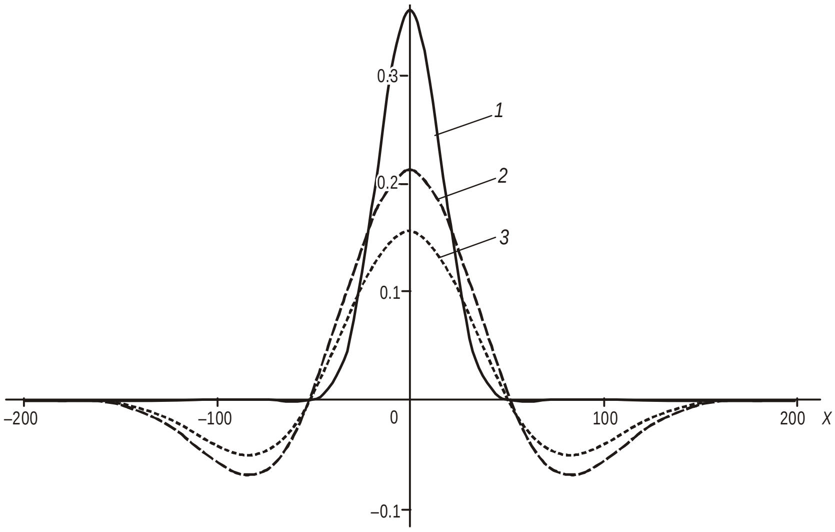

Fig.2. The dependence of vertical displacements of the Earth's surface on the horizontal coordinate at different times for the case if the width of the region of the initial displacement l is 10 km. Curve 1 corresponds to 30 years, curve 2 to 300 years, curve 3 to 1000 years

If the lithosphere underlying the elastic upper crust had a rheology of Newtonian fluid with viscosity η, the analog of the shear modulus $G_A^*(s)$ would have to be replaced by ηs, and equation (24) would be written as

$$U_z^* = U_0(k) \cdot \frac{2|k|\eta}{\rho g + Nk^4 + 2|k|\eta s} \ . \tag{61}$$

The inversion of the Laplace image (61) gives

$$U_z(t) = U_0(k) \exp\left(-\frac{\rho g + Nk^4}{2|k|\eta} t \right), \tag{62}$$

The relation (62) is valid for any times t, unlike the relation (30) that characterizes the Andrade medium and is valid only for not too long times limited by the condition (29). Comparison of equalities (62) and (30) shows that for not too long times the Andrade medium can be characterized by an effective viscosity

$$\eta_{\text{eff}} = At^{2/3} \, . \tag{63}$$

As follows from (63), for times of the order of 1000 years $\approx 3 \cdot 10^{10}$ s typical for small-scale postglacial flows, the typical value $A = 5 \cdot 10^{12}$ Pa·s$^{1/3}$ of Andrade rheological parameter corresponds to an effective viscosity $\eta_{\text{eff}} \approx 5 \cdot 10^{19}$ Pa·s. Such an estimate is consistent with an estimate of the viscosity obtained in the study of small-scale postglacial flows within the rheological model of the Newtonian fluid [*Cathles*, 1975].

At long times, the displacement of the surface of the lithosphere with the Andrade rheology is fundamentally different from the displacements that arise at the Newtonian rheology of the lithosphere. In the neighborhood of the point $s = 0$, the right-hand side of (24) can be represented as a series

$$U_z^*(s) = U_0(k)\left(\frac{s^{-2/3}}{\Phi(k)} - \frac{s^{-1/3}}{\Phi^2(k)} + \frac{1}{\Phi^3(k)}\dots\right). \tag{64}$$

According to the theorem on the asymptotic behavior of the original [*Doetsch*, 1967], the Laplace original $u_z(t)$ for a long time can be represented in the form of a series whose terms are obtained as a result of the inverse Laplace transform of each term in the series (64). Retaining only the first term in the expansion, we find

$$U_z(t) = \frac{U_0(k)}{\Phi(k)}\frac{t^{-1/3}}{\Gamma(2/3)}. \tag{65}$$

The asymptotic dependence (65) is valid for long times, when

$$t >> \left(\Phi(k)\right)^{-3}. \tag{66}$$

As follows from (65), for large times in the lithosphere with Andrade rheology, modes with different wave numbers k decay in accordance with the same law $t^{-1/3}$; and the effect of propagation of displacements of the surface disappears, which occurs at shorter times. Since the minimum value of $\Phi(k)$ is reached for $k=k_m$, inequality (66) holds for any wave numbers if

$$t \gg \left(\Phi(k_m)\right)^{-3}, \qquad \left(\Phi(k_m)\right)^{-3} \approx 5 \cdot 10^{11}\text{s} \approx 2 \cdot 10^4\text{years}. \tag{67}$$

In the other limiting case (t<<$2{\cdot}10^4$ years) considered in this paper, modes with wave numbers $k \approx k_m$ undergo weakly attenuate and determine the values of displacements whereas modes with wave numbers, strongly different from k_m, decay according to the law $t^{-1/3}$ and make a small contribution to the displacement of the surface.

Relations (67) are valid for a small-scale initial loading, for which $l<2\pi/k_m$. In the case of a large-scale load, i.e., when $l>2\pi/k_m$, the asymptotics, at which surface displacement occurs according to the law $t^{-1/3}$, becomes applicable at shorter times than with a small-scale load. In this case, the propagation of displacements from the region of the initial surface perturbation occurs only at small times of the order of several tens of years, and later the displacements do not propagate, but simply gradually decay.

Conclusions

Plate tectonics allows only small deformations in lithospheric plates, and therefore the creep of the lithosphere is transient. Assuming a steady-state creep of the lithosphere described by a power non-Newtonian rheological model, the effective viscosity of the lithosphere is very large and the lithosphere behaves like an elastic medium. Transient creep of the lithosphere greatly reduces its effective viscosity, and only a thin layer of the upper crust exhibits elasticity. The lithosphere, i.e. the cold boundary layer formed by mantle convection includes not only the elastic crust, but also the asthenosphere corresponding to data on postglacial isostatic recovery flows.

Numerous studies, where the mantle postglacial flows are considered within the frames of the Newtonian rheological model determine an effective viscosity corresponding to processes with duration of the order of several thousand years. The effective viscosities thus obtained cannot characterize much slower geological processes. Transient creep of the lithosphere is described by the linear Andrade model. The rheological parameter Andrade characterizes low-strain processes of any duration with very different effective viscosities.

To estimate the effective viscosity, the initial distribution of the vertical displacements is Fourier transformed and the time dependence of the mode with a fixed value of the wave number k is investigated. In order to obtain the dependence of the displacement on the horizontal coordinate, it is necessary to perform an inverse Fourier transform that allows detecting the perturbation propagation from its initial point along the Earth's surface. This propagation that can be called the inertialess Rayleigh wave is due to the fact that modes with different wave numbers k attenuate with different rates. If vertical displacements have occurred in some region of the Earth's surface and have violated the isostatic equilibrium of the crust, then the process of isostatic recovery is not reduced only to a gradual decrease in the initial displacements in this region. The displacements propagate beyond the initial region, as can be seen in Fig. 1 and 2. This propagation (the inertialess Rayleigh wave) occurs with strong attenuation.

The process under consideration is related to the Elsasser diffusion waves [*Elsasser*, 1969] that are excited by initial horizontal displacements. In the Elsasser model, it is assumed that the horizontal flow of the Couette is realized in the asthenosphere, and the flow velocities vanish at the base of the asthenosphere. This artificial boundary condition was also used in subsequent studies [*Turcotte, Schubert*, 1985]. In the present work, the investigation of diffusion waves is carried out within the frames of the problem formulation typical for surface waves: the asthenosphere is modeled as a half-space, and displacements are bounded at large depths. Therefore, since the displacements are vertical at the upper horizontal boundary, the diffusion waves under consideration can be called the inertialess Rayleigh waves. If we consider horizontal initial displacements at the upper boundary and simulate the asthenosphere as a half-space, then the emerging diffusion waves can be called inertialess Love waves [*Birger*, 1989].

The initial displacement may be caused by a depression formed after a melted glacier or a dried salt lake, or by a rise formed, for example, in the eruption of a volcano. For several thousand recent years, vertical displacements that violate isostasy occurred in different regions of the Earth's surface. The inertialess Rayleigh waves generated by these initial displacements propagate throughout the Earth's surface and can be considered as the mechanism of modern vertical movements of the earth's crust. These modern movements that occur during the last several thousand years are well observed, and their speed is estimated as $1 \div 10$ mm / year. The maximum speed of modern movement (1 cm / year) is seen on the northern coast of the Gulf of Bothnia and is caused by the postglacial uplift of Fennoscandia.

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